Physics 523, Quantum Field Theory II
Midterm Examination

Due Monday, 29th March 2004

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1. Loop Integrals in Dimensional Regularization

We are to verify the identity

$$\int \frac{d^dq}{(2\pi)^d} \frac{(d-2n)q^2 - dm^2}{(q^2-m^2)^{n+1}} = 0.$$

Noting the results of homework 6 and the elementary properties of the Γ-function, we may proceed directly.

$$\begin{aligned}
\int \frac{d^dq}{(2\pi)^d} \frac{(d-2n)q^2 - dm^2}{(q^2-m^2)^{n+1}} &= (d-2n)\frac{(-1)^n i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n+1)}\frac{1}{(m^2)^{n-d/2}} - dm^2\frac{(-1)^{n+1}i}{(4\pi)^{d/2}}\frac{\left(n+1-\frac{d}{2}\right)}{\Gamma(n+1)}\frac{1}{(m^2)^{n+1-d/2}},\\
&= \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{d}{\Gamma(n+1)}\left[(d/2-n)\frac{\Gamma\left(n-\frac{d}{2}\right)}{(m^2)^{n-d/2}} + m^2\frac{\Gamma\left(n+1-\frac{d}{2}\right)}{(m^2)^{n+1-d/2}}\right],\\
&= \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{d}{\Gamma(n+1)}\frac{1}{(m^2)^{n-d/2}}\left[-(n-d/2)\Gamma(n-d/2) + \Gamma(n+1-d/2)\right],\\
&= \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{d}{\Gamma(n+1)}\frac{1}{(m^2)^{n-d/2}}\left[-\Gamma(n+1-d/2) + \Gamma(n+1-d/2)\right],\\
&= 0.
\end{aligned}$$

$$\boxed{\therefore \int \frac{d^dq}{(2\pi)^d} \frac{(d-2n)q^2 - dm^2}{(q^2-m^2)^{n+1}} = 0.} \tag{1.a}$$

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Let us now evaluate the following loop integral,

$$I(p^2, m_1^2, m_2^2) = -ie^2 \int \frac{d^dq}{(2\pi)^d} \frac{1}{((q+p/2)^2 - m_1^2 + i\epsilon)((q-p/2)^2 - m_2^2 + i\epsilon)}.$$

To evaluate this integral lucidly, let us first introduce the change of variables $k \equiv q + p/2$. Introducing the Feynman parameter x, the integral becomes,

$$\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{[x((k-p)^2 - m_2^2 + i\epsilon) + (1-x)(k^2 - m_1^2 + i\epsilon)]^2}.$$

Introducing the variables,

$$\ell \equiv k - xp \qquad \text{and} \qquad \Delta \equiv x(x-1)p^2 + xm_2^2 + (1-x)m_1^2,$$

we see that

$$\begin{aligned}
I(p^2, m_1^2, m_2^2) &= \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2},\\
&= \int_0^1 dx \left[\frac{i}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}\right],\\
&\underset{d\to 4}{\sim} \frac{i}{(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} - \log\Delta - \gamma_E + \log(4\pi) + \mathcal{O}(\epsilon)\right]
\end{aligned}$$

$$\boxed{\therefore I(p^2, m_1^2, m_2^2) \underset{d\to 4}{\sim} \frac{i}{(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} + \log\frac{1}{x(x-1)p^2 + xm_2^2 + (1-x)m_1^2} - \gamma_E + \log(4\pi)\right].} \tag{1.b}$$

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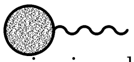
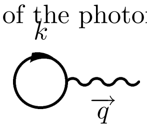
The One-Loop Structure of Quantum Electrodynamics

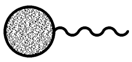
While studying the superficial divergences of quantum electrodynamics, we noted that gauge invariance–and hence the Ward identity–made several superficially divergent diagrams either converge or vanish. We are to verify these claims explicitly.

Superficially, the one-point function of the photon has a cubic divergence. Let us demonstrate that in fact, to the one-loop order, the one-point function of the photon vanishes.

To one-loop order, we see that  $=$ 

The amplitude for the above diagram is given by

$$\begin{aligned}
i\mathscr{M} &= (-1)\epsilon^*_\mu(q)\int\frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[\frac{i(\not k+m_e)}{(k^2-m_e^2+i\epsilon)}(-ie\gamma^\mu)\right],\\
&= -\epsilon^*_\mu(q)e\int\frac{d^dk}{(4\pi)^d}\frac{\mathrm{Tr}\,(\not k\gamma^\mu+m\gamma^\mu)}{(k^2-m_e^2+i\epsilon)},\\
&= -\epsilon^*_\mu(q)4e\int\frac{d^dk}{(4\pi)^d}\frac{k^\mu}{(k^2-m_e^2+i\epsilon)},\\
&= 0.
\end{aligned}$$

Therefore, to one-loop order,  $= 0.$

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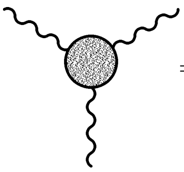
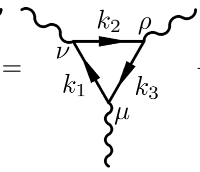
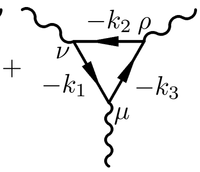
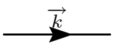
Similarly, we argued that although the photon three-point function has a superficial, linear divergence, its amplitude should also vanish. Let us now demonstrate this fact.

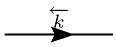
To one-loop order, we see that

 $=$ 

$+$ 

Note that the second diagram has been labeled the same as the first diagram but with relative minus signs on the momenta k. This is because the Feynman propagator has the property that



$= \dfrac{i(\not k+m_e)}{(k^2-m_e^2+i\epsilon)}$ whereas 

$= \dfrac{i(-\not k+m_e)}{(k^2-m_e^2+i\epsilon)}.$

Let us consider the evaluation of the first diagram. Its amplitude is proportional to integration over

$$\mathrm{Tr}\left[\gamma^\mu(\not k_1+m_e)\gamma^\nu(\not k_2+m_e)\gamma^\rho(\not k_3+m_e)\right].$$

Because only those traces over an even number of γ-matrices are non-vanishing, this is equal to

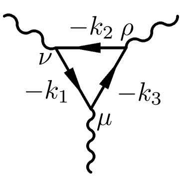
$$\mathrm{Tr}\left[\gamma^\mu\not k_1\gamma^\nu\not k_2\gamma^\rho\not k_3\right]+m_e^2\left(\mathrm{Tr}\left[\gamma^\mu\not k_1\gamma^\nu\gamma^\rho\right]+\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\not k_2\gamma^\rho\right]\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\not k_3\right]\right).$$

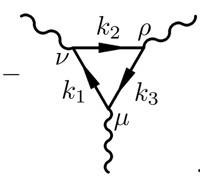
Notice that the only remaining traces involve an odd number of momenta k.

Similarly, we see that the amplitude of the second diagram is proportional to integration over

$$\mathrm{Tr}\left[(-\not k_3+m_e)\gamma^\rho(-\not k_2+m_e)\gamma^\nu(-\not k_1+m_e)\gamma^\mu\right]=-\mathrm{Tr}\left[\not k_3\gamma^\rho\not k_2\gamma^\nu\not k_1\gamma^\mu\right]-m_e^2\left(\mathrm{Tr}\left[\not k_3\gamma^\rho\gamma^\nu\gamma^\mu\right]+\mathrm{Tr}\left[\gamma^\rho\not k_2\gamma^\nu\gamma^\mu\right]\mathrm{Tr}\left[\gamma^\rho\gamma^\nu\not k_1\gamma^\mu\right]\right).$$

But, noting identity (5.7) of Peskin and Schroeder, the traces of each expression are equal. Therefore, the negative contribution from the second diagram cancels the contribution from the first.



$= -$ 

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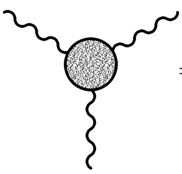
Therefore, to one-loop order,  $= 0.$

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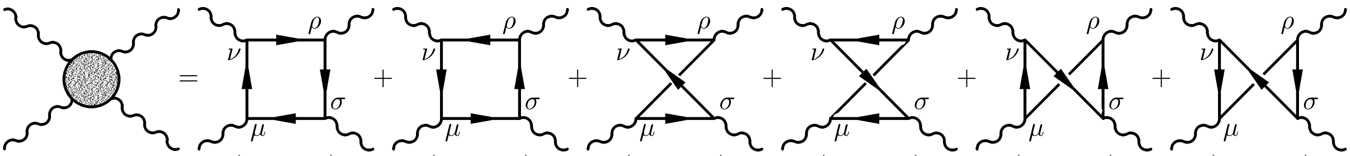
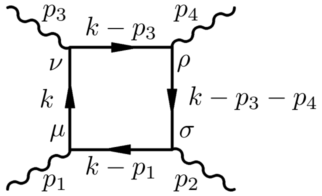
Lastly, our analysis showed that the photon four-point function has a logarithmic, superficial divergence, but by gauge invariance this amplitude is convergent. We are to demonstrate that the photon four-point function does not diverge to the one-loop order in perturbation theory.

To one-loop order, we see that



Because it is our task to demonstrate that the above amplitude converges–rather than actually compute the amplitude–we may make several helpful simplifications. To illustrate the first major simplification, let us analyze the first diagram, (I).



$$= -e^4 \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left[(\not{k} - \not{p}_1 + m_e)\gamma^\mu(\not{k} + m_e)\gamma^\nu(\not{k} - \not{p}_3 + m_e)\gamma^\rho(\not{k} - \not{p}_3 - \not{p}_4 + m_e)\gamma^\sigma\right]}{((k-p_1)^2 - m_e^2)(k^2 - m_e^2)((k-p_3)^2 - m_e^2)((k-p_3-p_4)^2 - m_e^2)},$$
$$= -e^4 \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left[\not{k}\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma\right]}{(k^2 - m_e^2)^4} + \text{ finite terms.}$$

Therefore, we see that the divergent part of each diagram is a function of *only the order* of γ-matrices in the trace.

Now, we claim that the divergence of diagram (I) is the same as (II), (III)$\sim$(IV), and (V)$\sim$(VI). First, note that the relative change of sign for the loop momentum k between each pair will not change the divergence of the diagram because each involves only $k^4 = (-k)^4$. Secondly, the ordering of the vertices are precisely reversed for each pair and so by identity (5.7) of Peskin and Schroeder they are equal. Therefore the total divergence of these six diagrams will be twice that of (I), (III), and (V) alone.

Let us continue to compute the divergence of diagram (I) before illustrating the sum of all six diagrams. Because, as we will show, the sum of the diagrams will converge, we will continue without dimensional regularization.[1]

In our calculation below, we will repeatedly make use of γ-matrix algebra proved in homework (including that of semester I). Also, note our use of identity (A.42) from Peskin and Schroeder. Let us begin to evaluate the divergence of diagram (I). The integrand is proportional to

$$\begin{aligned}
\mathrm{Tr}\left[\not{k}\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma\right] &= k_\alpha k_\beta k_\gamma k_\delta \mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\gamma\gamma^\rho\gamma^\delta\gamma^\sigma\right], \\
&\to \frac{1}{d(d+2)}(k^2)^2\left(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}\right)\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\gamma\gamma^\rho\gamma^\delta\gamma^\sigma\right], \\
&\propto \mathrm{Tr}[\gamma\gamma^\mu\gamma\,\gamma^\nu\,\gamma\gamma^\rho\gamma\,\gamma^\sigma] + \mathrm{Tr}[\gamma\gamma^\mu\gamma\gamma^\nu\gamma\gamma^\rho\gamma\,\gamma^\sigma] + \mathrm{Tr}[\gamma\gamma^\mu\gamma\gamma^\nu\gamma\gamma^\rho\gamma\,\gamma^\sigma], \\
&= \mathrm{Tr}[(-2\gamma^\mu)\gamma^\nu(-2\gamma^\rho)\gamma^\sigma] + \mathrm{Tr}[(-2)\gamma^\nu\gamma\gamma^\mu\gamma^\rho\gamma\,\gamma^\sigma] + \mathrm{Tr}[\gamma\gamma^\mu(-2\gamma^\nu)\gamma^\rho\gamma\,\gamma^\sigma], \\
&= 4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] - 2\mathrm{Tr}[\gamma^\nu 4g^{\mu\rho}\gamma^\sigma] - 2\mathrm{Tr}[-2\gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma], \\
&= 8\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] - 8g^{\mu\rho}\mathrm{Tr}[\gamma^\nu\gamma^\sigma], \\
&= 32\left(g^{\rho\sigma}g^{\mu\nu} - g^{\nu\sigma}g^{\mu\rho} + g^{\mu\sigma}g^{\nu\rho}\right) - 32g^{\mu\rho}g^{\nu\sigma}, \\
&\propto \left(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right).
\end{aligned}$$

Therefore, when we evaluate the amplitude for all six diagrams, the divergent integral will be over a term proportional to $(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$ together with the analogous terms under the other two distinct permutations. Therefore, the amplitude's divergence will be proportional to,

$$(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) + (g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu}) + (g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}) = 0.$$

Therefore, the photon's four-point function is convergent to loop-order in QED. ὅπερ ἔδει δεῖξαι

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[1]It is easier for our purposes to work with $d = 4$ trace-algebra. Because the total divergence will vanish in $d = 4$, it must also vanish in general dimensional regularization.

The β-function of Quantum Chromodynamics

We are given that, at one-loop order in perturbation theory, the divergent parts of the counter terms of quantum chromodynamics are

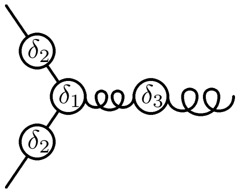
$$\delta_1 = -\frac{7}{2}\frac{g^2}{(4\pi)^2}\log\frac{\Lambda^2}{M^2}, \quad \delta_2 = -\frac{1}{2}\frac{g^2}{(4\pi)^2}\log\frac{\Lambda^2}{M^2}, \quad \text{and} \quad \delta_3 = \left(5-\frac{2}{3}n_f\right)\frac{g^2}{(4\pi)^2}\log\frac{\Lambda^2}{M^2},$$

where the δ_i are defined in analogy to quantum electrodynamics. We see that these directly imply that

$$B_g = \frac{7}{2}\frac{g^2}{(4\pi)^2}, \quad A_f = -\frac{1}{2}\frac{g^2}{(4\pi)^2}, \quad \text{and} \quad A_{gl} = \left(5-\frac{2}{3}n_f\right)\frac{g^2}{(4\pi)^2},$$

where A_f corresponds to fermion self-energy and A_{gl} corresponds to gluon self-energy.

Let us now compute the β-function for the strong coupling g. This corresponds to the diagram,



Therefore, because $\beta_g = -2gB_g - 2gA_F - gA_{gl}$, we see that

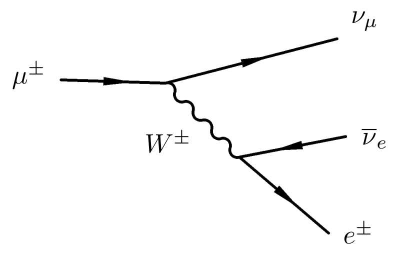
$$\boxed{\therefore \beta_g = -\left(11-\frac{2}{3}n_f\right)\frac{g^3}{16\pi^2}.} \tag{3.a}$$

In homework 10, we computed the general running coupling constat associated with quantum chromodynamics. To relate that result with our work here, we should set the undetermined constant β_1 to $\left(11-\frac{2}{3}n_f\right)$. So from our results of homework 10, we see that the square of the running coupling $\overline{g}$ is

$$\boxed{\therefore \overline{g}^2 = \frac{g^2}{1+\frac{g^2}{8\pi^2}\left(11-\frac{2}{3}n_f\right)\log(p/M)}.} \tag{3.b}$$

We see that the coupling constant will be asymptotically free if $11 > 2/3n_f$. This is because asymptotic freedom is directly a result of a negative β-function. It is clear that $\beta_g < 0$ only if $n_f < 33/2 = 16.5$. Also, again by the results of homework 10, we see that at large energy ($p/M \to \infty$), the square of the coupling constant can be approximated by

$$\boxed{\overline{g}^2 \underset{\frac{p}{M}\to\infty}{\approx} \frac{8\pi^2}{\left(11-\frac{2}{3}n_f\right)\log(p/M)}.} \tag{3.c}$$


ν_μ
$\mu^\pm$
$W^\pm$
$\overline{\nu}_e$
$e^\pm$